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# Revisiting Characterization of Semigroups: Exploring limitations of Independence

***[1]Enyiduru, E. H., [2]Efiong, F.A., [3]Effiong, L. E., [4]Akak E. O., [5]Mfon, A. E., [6]Sampson, M. I.**

[*1]Department of Mathematics, Michael Okpara University of Agriculture Umudike

[2]Department of Mathematics, University of Uyo, Uyo, Akwa Ibom State, Nigeria

[3]Department of Mathematics, Abia State Polytechnic, Aba, Abia State, Nigeria

[4]Department of Mathematics, Akwa Ibom State University, Ikot Akpaden, Nigeria

[5]Department of Mathematics, Akwa Ibom State University, Ikot Akpaden, Nigeria

[6]Department of Mathematics, Akwa Ibom State University, Ikot Akpaden, Nigeria

[*]Corresponding Author Email: hannahekwomchi173@gmail.com, Tel: +2347060660000

**Abstract**

This paper reviews the evaluation of the characterization of semigroups Independence, generating systems by Sampson, M. I. (see [2]), which began with studying the work of Howie and Ribeiro (see [1]) to investigate the effect of the concept of independence on semigroup generating systems. While independence is often considered fundamental in determining minimal generators for semigroups, we show instances where this notion proves inadequate. Specifically, with illustrated variations in the role of independence in Semigroup as compared to vector spaces. The study shows reliable characterization of independent generating sets that helps in the selection algorithm for the basis elements of any given semigroups.

**Keywords:** Semigroups, Rank characterization, Independence, Counterexamples, Commutative semigroups, Non-commutative semigroups

## 1. Introduction

Blockchain technology has emerged as a transformative force in various industries, Semigroups, algebraic structures defined by a binary operation satisfying associativity, play a significant role in various areas of mathematics, including algebra, combinatorics, and theoretical computer science. One key aspect in the study of semigroups is the characterization of their rank, which measures the structural complexity of these mathematical objects.

The concept of rank in semigroups serves as a fundamental indicator of the size of the smallest generating set necessary to produce all elements of the semigroup. Understanding the rank of a semigroup provides insights into its algebraic properties, its structure, and its computational complexity.

One approach to characterizing the rank of a semigroup is based on the notion of independence. Independence refers to subsets of elements within a semigroup that cannot be expressed as products of other elements in the subset. The significance of independence in rank determination lies in its ability to identify minimal generating sets, or bases, that efficiently generate the entire semigroup.

Historically, independence-based characterizations of rank have been widely studied and applied in the literature. Howie and Ribeiro (see [1]) proposed a specific characterization of rank in semigroups based on independence, which has been influential in the field. This characterization suggests that the basis elements of a semigroup should correspond to the maximal independent subsets.

However, despite the apparent significance of independence in rank determination, there are cases where this concept fails to provide an accurate characterization of rank. In certain semigroups, the existence of finite bases coexists with the presence of a larger number of independent elements than expected based on conventional assumptions. These counterexamples challenge the traditional understanding of the relationship between independence and rank in semigroups.

In this paper, we delve into the intricacies of rank characterization in semigroups, focusing on the role of independence and its limitations. By examining specific instances where independence-based characterizations falter, we aim to shed light on alternative approaches to rank characterization and contribute to a deeper understanding of semigroup structures.

## 2. PRELIMINARIES

In this section, we provide formal definitions of key concepts related to semigroups and rank characterization. Definition 2.1 (Semigroup). Let $S$ be a non-empty set "°" be a binary operation on S. The pair ($S$, °) is a semigroup if, for all $a$, $b$, $c \in S$, the operation ° satisfies the associative property:

$$(a°b)°c = a°(b°c).$$

**Example 2.2.** Consider the set $S$ = {1,2,3,4} with the binary operation ° defined as $addition$ $modulo$ 5. Then, ($S$, °) forms a semigroup since $addition$ $modulo$ 5 is associative:

• (1 + 2) + 3 = 3 + 3 = 6 ≡ 1($mod$5)

• 1 + (2 + 3) = 1 + 5 ≡ 1($mod$5)

Thus, the pair ($S$, °) satisfies the associative property, fulfilling the definition of a semigroup.

**Definition 2.3 (Rank).** Let $S$ be a semigroup and $A = \{a_1, a_2, \dots, a_n\} \subset S$. The rank, ($S$), of $S$ is the smallest non-negative integer $n$ such that for any $x_j \in S,\ j \in N$,

$$x_j = \prod_{i=1}^{n} a_i, \ \ \forall i \in N$$

,

**Example 2.4 [3].** Consider the five element Brandt Semigroup,

$B_2 = \{a, b, ab, ba, o \mid aba = a, bab = b, \ a_2 = b_2 = 0\}$

The multiplication is done on the rule below governing Brandt semigroup:

$$(i, a, j)(k, b, l) = \begin{cases} (i, ab, l) \ if \ j = k \\ 0, \quad otherwise \end{cases}$$

The minimal generating subset of $B_2$ is $\{a, b\}$ and rank $(B_2)$ = 2. The product of $B_2 \times B_2 \times B_2$ above is

$\{a, b, ab, ba, o\} \times \{a, b, ab, ba, o\} \times \{a, b, ab, ba, o\}$ =$\{(a, a), (a, b), (a, ab), (a, ba), (a, 0), (b, a), (b, b), (b, ab), (b, ba), (b, 0), (ab, a), (ab, b), (ab, ab), (ab, ba), (ab, 0), (ba, a), (ba, b), (ba, ab), (ba, ba), (ba, 0), (0, a), (0, b), (0, ab), (0, ba), (0, 0)\} \times \{a, b, ab, ba, o\}$

= $\{((a, a), a), ((a, a), b), ((a, a), ab), ((a, a), ba), ((a, a), 0), ((a, b), a), ((a, b), b), ((a, b), ab), ((a, b), ba), ((a, b), 0), ((a, ab), a), ((a, ab), b), ((a, ab), ab), ((a, ab), ba), ((a, ab), 0), ((a, ba), a), ((a, ba), b), ((a, ba), ab), ((a, ba), ba), ((a, ba), 0), ((a, 0), a), ((a, 0), b), ((a, 0), ab), ((a, 0), ba), ((a, 0), 0), \cdots, ((0, 0), 0)\}$.

Order of $(B_2 \times B_2 \times B_2)$ =125.

Minimal generator = $\{((a, a), a), ((b, b), b)\}$ which has 2 as the rank.

Rank $(B_2 \times B_2 \times B_2) = 2 \leq$ Rank $(B_2)$ + Rank $(B_2)$ + Rank $(B_2) < 2 + 2 + 2 = 6$.

**Remark 2.4.1.** The rank of the direct product of any algebraic system depends on the structure of the system.

**Definition 2.5 (Basis).** Let $S$ be a semigroup. A subset $B \subset S$ is a basis of S if it satisfies the following two conditions:

1. B is a generating set for $S$, i.e., every element of $S$ can be expressed as a product of elements

from $B$.

2. $B$ is minimal in cardinality among all generating sets for $S$, i.e., there is no proper subset

of $B$ that is also a generating set for $S$.

**Example 2.6 [13].** Consider the six elements semigroup $S$, which is a 2 × 2 real matrix under matrix multiplication,

$o = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, ab = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, ba = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Table 1. Multiplication table:

| | 0 | $i$ | $a$ | $b$ | $ab$ | $ba$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $i$ | 0 | $i$ | $a$ | $b$ | $ab$ | $ba$ |
| $a$ | 0 | $a$ | 0 | $ab$ | 0 | $a$ |
| $b$ | 0 | $b$ | $ba$ | 0 | $b$ | 0 |
| $ab$ | 0 | $ab$ | $a$ | 0 | $ab$ | 0 |
| $ba$ | 0 | $ba$ | 0 | $b$ | 0 | $ba$ |

Let $U = \{i, a, b\} \subseteq S$. Notice that every element from $\{o, i, a, b, ab, ba\}$ can be expressed as product of elements of $U$ since the elements $i, a, b \in U$ multiply each other to produce $S = \{o, i, a, b, ab, ba\}$ .
Notice also that there is no other proper subset of elements, say, $A \subset U$ in $S$ that generates $S$: Therefore $U$ is a minimal generating set and by definition 2.5, the basis in $S$.

**Remark 2.6.1. (Notation).** We denote the subsemigroup generated by $a$ as $\langle a \rangle$.

**Definition 2.7 (Independence).** Let $S$ be a semigroup. A subset $I \subset S$ is independent if, for all distinct elements $ai \in I$, each $ai$ fails to exist in the subsemigroup $\langle (S \backslash ai, \circ) \rangle$.

**Example 2.8 (Independence).** Consider the semigroup $S$ defined in 2.6. The elements $i$, $a$ and $b$ are independent elements:
If we take out $i$ from $U$, notice that $\{i\}$ is excluded from the subsemigroup generated by the remaining elements $U \setminus \{i\}$ of $S$. That is, $\{i\}$ is excluded from $\langle U \setminus \{i\} \rangle = \{o, a, b, ab, ba\}$.
Notice also that $\{a\}$ $and$ $\{b\}$ are excluded from the subsemigroup
$\langle U \setminus \{a\} \rangle = \langle \{o, i, b\} \rangle = \{o, i, b\}$ $and$ $\langle U \setminus \{b\} \rangle = \langle \{o, i, a\} \rangle = \{o, i, a\}$ respectively.
We conclude that the subset $U$ is independent.

**Remark 2.9.** These definitions lay the groundwork for our subsequent analysis of rank characterization in semigroups, particularly focusing on the role of independence in this context.

## 3.0. CERTAIN CHARACTERIZATIONS ABOUT INDEPENDENCE IN VECTOR SPACES AND THEIR EXPECTED ANALOGUE IN SEMIGROUP

**Definition 3.1.** Let the set ( ) be the power-set of the non-empty set $W$. A set $A \in (W)$ with a property $pr$ is maximal (minimal) if $\nexists U \in (W)$ with the property $pr$ such that $A \subset U$ $(U \subset A)$ and $A \neq U$.

Let $\{V, +,*\}$ be a vector space over a field $F$ and let $H \subset V$ . The statements of the following theorem are popularly known to be true about independence and generating systems of vector spaces.

**Theorem 3.2.**

1. A maximal independent set $B \subset V$ is a basis;

2. A minimal generating set B ⊂ V is a basis i.e. it is independent;

3. If $H \subset V$ is an independent set and $v \in V$ is independent from $H$ then $H \cup \{v\}$ is an independent set.

Equivalently any independent set has an extension to a basis.

4. In a vector space $\{V, +,*\}$ over a field $F$ the cardinality of all maximal independent sets is the same,

which is the dimension of a vector space.

The proof is left for the reader.

**3.2.1. Basis in Vector Spaces.**

The algorithm to select a basis in vector spaces is based on the properties of independence listed in Theorem

3.2. We give the algorithm below.

**3.2.2. Algorithm for Vector Space Generating set [].**

Let $\{V, +,*\}$ be a vector space over a field $F$ and let $B \subset V$ hold the selected basis elements, let $B = \emptyset$

for starting. Let $G \subset V$ be the starting system of vectors. Then

(a) If $G \neq \emptyset$ then $\exists x \in G$ otherwise [exit];

(b) If $x \notin (B)$ then $B = B \cup \{x\}\ and\ G = G \setminus (B)$ otherwise $G = G \setminus \{x\}$.

(c) goto (a).

This algorithm collects a basis in $B$ by the theorem 3.2 (1) and (3). Since these ((1) and (3)) are not necessarily true in semigroups (as we shall soon see in section 3.3), this cannot be adopted to semigroups.

One most important consideration given in 3.3 is that a maximal independent subset of a semigroup may not be a generating set and a basis must be a generating set. Since a generating set may not be reached from an independent set, we will work from starting with a generating set and we will sustain it throughout the process. Note that if $\{S,*\}$ is a semigroup then $S \subset S =< S >$ is a generating set. Therefore $G = S$ always can serve as a starting generating set.

## 3.3. INDEPENDENCE IN THE CONSTRUCTION OF GENERATING SETS IN SEMIGROUP

We start with some relevant definitions.

**Definition 3.3.1.** The set $H \subset V$ is a generating set if $L(H) =: < H > = V$ .

**Definition 3.3.2.** Let $\emptyset \neq H \subset V$ then $x \in V$ is independent from $H$ if $x \notin L(H)$.

**Definition 3.3.3.** The set $H \subset V$ is an independent set if $x \notin (H \setminus \{x\}) = < H \setminus \{x\}) >, \forall x \in H$.

**Definition 3.3.4.** An independent generating set $B \subset V$ is called a basis.

**Remark 3.3.5.** It can be shown with examples (see Example 3.3.7) that handling generating sets and independence including construction of bases in semigroup theory requires a technique different from the technique used in vector spaces. Important conclusions are the deviations from the properties discussed in theorem 3.2 about vector spaces stated in theorem 3.3.9 about semigroups.

1. Independence: There may exist maximal independent set of elements in a semigroup which does not form a generating set.

2. There may exists independent set in a semigroup which cannot be expanded to a generating set 3.

Adding an element independent from an independent set may make the independent set dependent in a semigroup.

### 3.3.6. CONSTRUCTION OF GENERATING SETS AND BASES IN SEMIGROUPS.

**Example 3.3.7.** Consider the following semigroup $\{S,*\}$ where $S := \{a, b, c\}$, and the operation is defined by the operation table (Table 1.):

Table 2: The operation table of $\{S,*\}$.

| $*$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $c$ | $c$ |
| $b$ | $c$ | $b$ | $c$ |
| $c$ | $c$ | $c$ | $c$ |

**Theorem 3.3.8.** The structure (S, $*$) is a semigroup, the operation is associative.

Proof. It suffices to check that associativity holds for $*$ in $(S,*)$: $a * (b *$ c) = $(a *$ b) $*$ c = $c$,

$a * (c *$ b) = $(a *$ c) $*$ b = $c$, $b * (a * c) = (b * a) * c = c$, $b * (c * a) = (b * c) * a = c$, $c * (b *$ a) =

$(c *$ b) $*$ a = $c$, $c * (a * b) = (c * a) * b = c$.

**Theorem 3.3.9.** In the semigroup $\{S,*\}$ given in Example 1.8, the following are true.

(1). The subset $\{b, c\} \subset S$ is a subsemigroup of $S$. The members of $\{b, c\}$ are independent:

$\{b, c\} \setminus \{b\} = \{c\}$ and $b \notin \langle\{b, c\}\setminus\{b\}\rangle = \langle\{c\}\rangle = \{c\}$. Similarly, $\langle\{b, c\}\setminus\{c\}\rangle = \langle\{b\}\rangle$ and $c$ $\notin$

$\langle\{b, c\}\{c\}\rangle = \langle\{b\}\rangle = \{b\}$.

(2). The subset $\{a, b\} \subset S$ is a generating set of $S$. Next, $a$ and $b$ are independent: We saw in (1) above that $b$ is independent. Notice also that $a \notin \langle\{a, b\}\setminus\{a\}\rangle = \langle\{b\}\rangle = \{b\}$. Hence $a$ and $b$ are independent and $\{a, b\}$ is a basis and $a \in S$ is independent from $\{b, c\} \subset S$ since this set is a

subsemigroup of $S$ and $a$ is not in it (the subsemigroup generated by $\{b, c\}$).

(3). Since $\{b, c\} \subset S$ is an independent subsemigroup, there is only one element in $S$ which is independent from this set, it is $a \in S$ and adding $a$ to $\{b, c\}$ forms a dependent set:

$\{a, b, c\} \setminus \{c\} = \{a, b\} \subset S\ gives\ \langle\{a, b\}\rangle = S \ni c.$

Proof. We proved the statements of the theorem in the theorem.

**3.3.10. Remarks.** The example shows that handling generating sets and independence including construction of bases in semigroup theory requires a technique different from the technique used in vector spaces. Important conclusions are the deviations from the properties discussed in theorem 3.2 about vector spaces now stated in theorem 3.4 about semigroups.

**Theorem 3.4.** In the semigroup $\{S,*\}$, the following are true

1. Independence: There may exist maximal independent set of elements in a semigroup which does not form a generating set.

2. There may exists independent set in a semigroup which cannot be expanded to a generating set.

3. Adding an element independent from an independent set may make the independent set dependent in a

semigroup.

**4.0. Independent Generating Sets of Semigroups: The Algorithm [14].**

The following is an algorithm for obtaining the basis set of any semigroup. We present some below.

**Theorem 4.1.** The algorithm below gives the basis of any semigroup $\{S,*\}$:

Let $\{S,*\}$ be a semigroup and let $\emptyset \neq G \subset S$ be a generating set. Let $S \supset B = \emptyset$.

(a) If $G \neq \emptyset$ then $\exists x \in G$ and $Gtest = G \setminus \{x\}$ otherwise [exit];

(b) If $x \notin (Gtest \cup B)$ then $B = B \cup \{x\}\ and\ G = G \setminus (B)$ otherwise $G = Gtest$;

(c) goto (a).

**4.1. Illustration.** At starting $G \cup B = G$ is a generating set. In step (a) $Gtest = G \setminus \{x\}$ removes x from $G$ hence the test $x \notin L(Gtest \cup B)$ in (b) checks for $x \notin L((G \cup B)\setminus\{x\})$ which checks if $x$ is independent from the generating set $G \cup B$.

Observe that in the case of vector spaces we check for the independence from B only.

In step (b) we change $B$ to a new value $Bnew = B \cup \{x\}$ and $Gnew = G \setminus (Bnew)$ if the independence of $x$ holds otherwise $Gnew = Gtest\ and\ Bnew = B$. The set $Gnew \cup Bnew$ is a generating set in the step (c): At step (a) ($G \cup B$) was a generating set at starting since $B = \emptyset$ and $G$ was a generating set.

After selecting $x \in G$ we have two options:

i). $x \notin ( \ \cup B)$ : Then $x \in Bnew(= B \cup \{x\})$ and $Gnew = Gtest$ hence $x \in G$ moves from $G$ into $B \subset S$. The final value of $G$ is $Gnew, = Gnew \setminus (Bnew)$.

The set $Gnew, \cup Bnew$ is a generating set: $(Gnew, \cup Bnew) \supset Gnew, \cup (Bnew) \supset Gnew \cup Bnew$.

Then $Gnew = G \setminus \{x\}$ and $Bnew = B \cup \{x\} \Rightarrow Gnew \cup Bnew = G \cup B$. Since $G \cup B$ was a generating

set and $L(Gnew,f \cup Bnew) \supset G \cup B$, $Gnew,f \cup Bnew$ is a generating set.

ii). $x \in ( \ \cup B)$ : Then $Gnew = Gtest$ and $Bnew = B$.

Since in this case $x \in ( \ \cup B)$ holds, $L( Gtest \cup B) \supset Gtest \cup B \cup \{x\} = G \cup B$. Hence

$Gnew \cup Bnew$ is a generating set.

$G \cup B \supset Gnew, \cup Bnew$ and $G = Gnew$, moreover if $x \in G$ is selected in step (a) then

$x \notin Gnew \supset Gnew$, hence or otherwise, any x can be selected only once. Therefore, if we select basis

vectors $x1, x2, . . . , xk, . ..$ one after the other then

$G0 := G$, $B0 := B = \emptyset$, $G1 := Gnew, , B1 := Bnew, . . ., Gk := (Gk-1)new,f \ , Bk := (Bk-1)new, . . ..$

Then the system of sets $\{GS, BS\}1 \leq s‘ < \infty$ and the selected points $\{xS\}1 \leq s \leq \infty$ fulfil:

$(Gk \cup Bk) \supset (Gk+1 \cup Bk+1)$; $Gk \supset Gk+1\ and\ Gk \neq Gk+1\ \forall\ 0 \leq k < \infty$

$xj \in Bj\ \forall\ 1 \leq j < k$, $Bk = \{xj \mid 1 \leq j \leq k\}$, $\forall\ k \in$ . (1)

**Theorem 4.2.** The set $Bk \subset S$ is independent $\forall 1 \leq k < \infty$.

Proof. The proof is Left for the reader

**Theorem 4.3 (See [14]).** In any semigroup $\{S,*\}$ there exists an independent generating set B $\subset$ S (basis). If in the semigroup $S$, $\exists x, y, z \in S$ such that $x * y = z$ and $z \notin \{x, y\}$, then there exists a basis $B \subset S$, $B \neq S$ ($B$ is a proper subset of $S$).

Proof. The existence follows from our reasoning above. If there exists x, y, z ∈ S as stated in the theorem

then let us select $z \in G = S$ as the first choice. Then $S\setminus\{z\} \supset \{x, y\} \Rightarrow x * y = z \in \langle\{x, y\}\rangle \subset \langle G\setminus\{z\} \ \rangle$ hence $z$ is not independent from $G\setminus\{z\}$ hence $z$ is not a selected basis element. Therefore, the selected basis $B$ is a proper subset of $S$.

**Remark 4.4.** The number of elements in the independent generating set of any given semigroup is the rank of the given semigroup

## 5.0. CONCLUSION

In any semigroup there exists a maximal independent subset that is a generating set. That subset is a basis.

With the example of the semigroup of integers in [13] and [5], unlike in vector spaces, it is proved that there are bases in a semigroup which can have different numbers of elements. A closely related situation is explained in [2]. With the example we showed that in a semigroup there may exists a maximal independent set of elements which is not a generating set.

The idea of the algorithm is as follows: When we select the next candidate to be a basis element from the current generating set then we assure that it is independent from all the previously selected basis elements (this is B) and it is independent from all possible future selections (this is the current $G$). This puts $x \notin (G \cup B)\setminus\{x\}$ into the selection condition.

**About the Autors:** The Authors do research in Algebra; Group theory, Computational Group theory, Algebraic Cryptography, Number theory and Mathematical Modelling.. See [2] – [14] for some of their work.

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